Sensitivity Analysis for Linear Estimands

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October 11, 2023

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Abstract

We propose a novel sensitivity analysis framework for linear estimands when identification failure can be viewed as seeing the wrong distribution of outcomes. Our family of assumptions bounds the density ratio between the observed and true conditional outcome distribution. This framework links naturally to selection models, generalizes existing assumptions for the Regression Discontinuity (RD) and Inverse Propensity Weighting (IPW) estimand, and provides a novel nonparametric perspective on violations of identification assumptions for ordinary least squares (OLS). Our sharp partial identification results extend existing results for IPW to cover other estimands and assumptions that allow even unbounded likelihood ratios, yielding a simple and unified characterization of bounds under assumptions like the c-dependence assumption of Masten and Poirier (2018). The sharp bounds can be written as a simple closed form moment of the data, the nuisance functions estimated in the primary analysis, and the conditional outcome quantile function. We find our method does well in simulations even when targeting a discontinuous and nearly infinite bound.

1 Introduction

Many important estimators in economics are observationally weighted averages of outcomes. Ordinary least squares (OLS) weights outcomes Y by $E[XX^T]^{-1}X$; regression discontinuity weights outcomes Y at a treatment discontinuity by their treatment assignment. These estimands have powerful interpretations under identifying restrictions like exogeneity or no manipulation. However, the identifying restrictions at the core of giving standard estimands meaningful interpretations often fail in empirical settings, and are often untestable. Even when these assumptions fail and treatment effects are confounded or agents sort across a treatment discontinuity, researchers often want to exploit the data to say something about the meaningful objects of interest. A large literature on sensitivity analysis has risen to the occasion (e.g., Masten and Poirier (2018); Gerard et al. (2020); Dorn and Guo (2022); Rambachan and Roth (2023)).

This paper provides a novel and tractable method for sensitivity analysis for linear estimators. The procedure in this paper nests Dorn and Guo (2022) and Masten and Poirier (2018) as special cases, and provides a new sensitivity analysis for difference-in-differences based on policy selection rather than functional form-dependent parallel trends failures (Rambachan and Roth, 2023). Our unified framework immediately

*This material is based upon work supported by the National Science Foundation Graduate Research Fellowship Program under Grant No. DGE-2039656. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

generates partial identification tools and sensitivity analyses in new domains: we immediately obtain novel Conditional Average Treatment Effect (CATE) bounds for regression discontinuity (RD) when there is sorting by potential outcomes.

Our procedure for sensitivity analysis requires making an assumption on the ratio of densities between the observed distribution and a counterfactual ("true") distribution where the identifying restriction holds. With this assumption, we can characterize sharp bounds in terms of optimization problems. Similar assumptions have been made in the treatment effect literature and interpreted as a limit on treatment selection (Tan, 2006; Masten and Poirier, 2018; Zhao et al., 2019; Dorn and Guo, 2022)

A further feature of our setup is that these optimization problems can be solved explicitly, so the sharp bounds can be written in closed form as a moment of the observed data. This feature makes the estimation straightforward to implement once we are able to write sensitivity analysis problems in terms of densities. We show our result naturally and simply extends to sensitivity assumptions that feature unbounded likelihood ratios, which is novel relative to the approaches in the existing literature that is typically limited to finite likelihood ratio bounds (Tan, 2022; Frauen et al., 2023) or Manski-type infinite likelihood ratio bounds (Gerard et al., 2020).

In the RD literature, we conduct sensitivity analysis to violations to the no-manipulation assumption on either side of the cutoff. McCrary (2008) propose an influential test for manipulation. We study what researchers can learn from the RD design when McCrary's test fails. Gerard et al. (2020) proposed a procedure to bound the CATE for the subpopulation that does not manipulate: a Conditional Local Average Treatment Effect (CLATE). A conditional local average treatment effect is likely to be less interesting to researchers than the original CATE object. Consequently, there is an open question of whether it is possible to partially identify the CATE. The CATE requires incorporating both manipulators and non-manipulators, which is challenging since we never observe non-manipulators directly.[1] This paper aims to fill this gap by showing how nontrivial bounds can be derived for the CATE. We partially identify the CATE by making a sensitivity assumption on the extent observations can manipulate given their potential outcomes. Our sensitivity assumption generalizes both Gerard et al. (2020)'s assumption and an exogeneity assumption that would allow researchers to interpret the RD estimand as a CATE and is always compatible with the identified manipulation from the McCrary test.

We build on the existing Inverse Propensity Weighting (IPW) literature by offering a simple framework to calculate valid and sharp bounds that also accommodate unbounded likelihood ratios. A literature began by Tan (2006), sparked by Zhao et al. (2019), and whose characterizations were sharpened by Dorn and Guo (2022), considers bounds on average treatment effect estimands under bounds on how much confounding can shift the odds of treatment, or equivalently the likelihood ratio between unobserved and observed outcomes. The IPW problem has then received much recent interest in the statistical literature (Bonvini et al., 2022; Tan, 2023; Huang et al., 2023; Zhang and Zhao, 2022; Soriano et al., 2021; Yin et al., 2022; Bruns-Smith and Zhou, 2023; Jesson et al., 2022). In an econometrics context, Masten and Poirier (2018) proposed similar bounds under an assumption bounding how much confounding can shift the probability of treatment. Tan (2022); Frauen et al. (2023) analyze a generalization of Tan (2006)'s model that excludes large values of Masten and Poirier (2018)'s sensitivity parameter. (Frauen et al. (2023) and Tan (2023) also study various longitudinal data generating processes.) We show that all values of Masten and Poirier (2018)'s sensitivity parameter fit within our even more general framework. A simple simulation under the c-dependence finds that a simple percentile bootstrap generates valid confidence intervals.

[1] We assume that manipulators only manipulate in one direction. For narrative clarity, we also assume that manipulators select in the direction of treatment.

In the OLS literature, we provide a sensitivity assumption in terms of likelihood ratio bounds. Cinelli and Hazlett (2020) provide a sensitivity assumption in OLS exclusion based on R^2 values and Rambachan and Roth (2023) provided a sensitivity assumption for Difference in Differences (DD) parallel trends based on slopes. These approaches often depend on the functional form of outcomes (e.g., Roth and Sant'Anna (2023)). Our likelihood ratio measure is invariant to monotonic transformations and so uses assumptions that researchers can form a view on across functional forms.

2 Framework for Sensitivity Analysis

2.1 General Setting

We observe data of an outcome Y and observable quantities R that do not include Y. These R could refer to regressors, for instance. We are interested in linear estimands that take the form

$$E_{True}\left[\lambda(R)Y\right] = \int \lambda(R)Y\,dP_{True} \tag{1}$$

where the integral is taken over some "true" distribution P_{True}, whose support is the same as the observed outcome. The true distribution is defined as the distribution where the model is correctly specified and the observable quantities R are drawn from the observed distribution of R. A prominent example is OLS. Suppose we are interested in the regression coefficient β of Y on R under the identifying restriction that $E_{True}[Y - R'\beta|R] = 0$. Then, $\lambda(R) = E[RR']^{-1}R$. Many other estimators commonplace in econometrics may also be written in this form: some examples explored in this paper include regression discontinuity designs, and inverse propensity weighting.

We observe data P_{Obs} of (R, Y) that may fail to satisfy the identifying restrictions. In the OLS case, we may have endogeneity in the sense that $E[X(Y - X'\beta)]$ is non-zero. We are interested in a distribution where the conditional mean of $Y \mid R$ is $X'\beta$, but endogeneity may lead to spurious conclusions about the parameter value.

We consider partial identification of meaningful parameters under bounded violations of the identifying restriction. We use a family of assumptions that can place meaningful bounds on our object of interest $E_{True}\left[\lambda(R)Y\right]$ without claiming to exactly observe the true estimand. The rest of this section provides a simple and precise way to calculate sharp bounds on this object.

We can often view identification failures as a failure to weight outcomes correctly. We first observe that the object of interest can be written as an expectation over the observable distribution.

$$E_{True}\left[\lambda(R)Y\right] = \int \lambda(R)Y\frac{dP_{True}}{dP_{Obs}}dP_{Obs} = E\left[W\lambda(R)Y\right], \tag{2}$$

where $W := dP_{True}/dP_{Obs}$ is assumed to exist and be finite (though not necessarily bounded). An identical way to frame the problem is that the object of interest is $E\left[\lambda_{True}(R)Y\right]$, where $\lambda_{True}(R) = W\lambda(R)$. If the support of $Y \mid R$ is the same under the observed and true distribution, then standard arguments imply $E\left[W|R\right] = 1$ almost surely.

We limit violations of the identifying restrictions in terms of the likelihood ratio between the true and observed conditional outcome distributions. In particular, we assume W satisfies $E\left[W|R\right] = 1$ and $W \in [\underline{w}(R), \bar{w}(R)]$ for some likelihood ratio bounding functions satisfying $0 \leq \underline{w}(R) \leq 1 \leq \bar{w}(R)$. These assumptions can be interpreted as the fact that the true distribution cannot be too different from the observed distribution, as $[\underline{w}(R), \bar{w}(R)]$ limit the extent of this difference. When $\underline{w}, \bar{w} = 1$, we recover the standard empirical assumption that $E_{True}[\lambda(R)Y] = E[\lambda(R)Y]$. As $\underline{w}$ and $\bar{w}$ get further from one, stronger identification failures are allowed.

The interpretation of our sensitivity assumption will be context-dependent. In the RD context, the likelihood ratio bounds can equivalently be phrased as bounds on the degree to which manipulation can be selected on potential outcomes. In the IPW context, the likelihood ratio bounds can be interpreted as bounds on the degree to which treatment is selection on potential outcomes. In the OLS context, the bounds are more opaque for interpretation.

The sharp bounds are the set of estimands that can be achieved by distributions in our class.

Definition 1. The sharp bounds in our general setting are the set of estimands $E_Q[\lambda(R)Y]$ under distributions Q satisfying (i) the distribution of R is the same under Q and the observed P_{Obs}, (ii) the support of $Y \mid R$ is the same under Q and the observed P_{Obs} , and (iii) $\frac{dQ}{dP_{Obs}} \in [\underline{w}(R), \bar{w}(R)]$ with probability one.

To find the upper and lower bounds of $E_{True}\left[\lambda(R)Y\right]$ under this setting, we solve

$$sup_W E\left[W\lambda(R)Y\right] s.t. W \in \left[\underline{w}\left(R\right), \bar{w}(R)\right], E[W|R] = 1 \tag{3}$$

This setup implies that the bound attained by Equation (3) must be sharp, because the upper bound can be arbitrarily well approximated by some W that is in the class of likelihood ratios that we allow in the model and cannot be exceeded by any W that is in the class of likelihood ratios we allow. The lower bound is written analogously by using inf in place of sup. This paper will focus on the supremum problem, as the infimum problem is entirely analogous. By solving the optimization problems, we obtain sharp bounds on the object of interest.

2.2 Identification Result

The general problem in Equation (3) turns out to have a simple closed-form solution. Let $Q_{\tau(R)}(\lambda(R)Y|R)$ denote the τth quantile of $\lambda(R)Y$ given R. When $\bar{w}(R)$ is finite and $P(\lambda(R)Y = Q_{\tau(R)}(\lambda(R)Y \mid R)) = 0$, the sharp bounds are:

$$W^*_{sup} = \begin{cases} \bar{w}(R) & if \quad \lambda(R)Y > Q_{\tau(R)}\left(\lambda(R)Y|R\right) \\ \underline{w}(R) & if \quad \lambda(R)Y \leq Q_{\tau(R)}\left(\lambda(R)Y|R\right) \end{cases}, \tau(R) = \frac{\bar{w}(R) - 1}{\bar{w}(R) - \underline{w}(R)} \tag{4}$$

$$W^*_{inf} = \begin{cases} \underline{w}(R) & if \quad \lambda(R)Y > Q_{\tau(R)}\left(\lambda(R)Y|R\right) \\ \bar{w}(R) & if \quad \lambda(R)Y \leq Q_{\tau(R)}\left(\lambda(R)Y|R\right) \end{cases}, \tau(R) = \frac{1 - \underline{w}(R)}{\bar{w}(R) - \underline{w}(R)} \tag{5}$$

Applying this formula to the formula $E_{True}[\lambda(R)Y] = E[W\lambda(R)Y]$ yields the following bound characterization.

Theorem 1. *When $\bar{w}(R)$ is finite and $P(\lambda(R)Y = Q_{\tau(R)}(\lambda(R)Y \mid R)) = 0$, the problem in (3) is solved by (4), and the analogous infimum problem is solved by (5). Hence, the upper bound is:*

$$E\left[W^*_{sup}\lambda(R)Y\right] = E\left[\lambda(R)Y\right] + E\left[\lambda(R)Y a\left(\underline{w}(R), \bar{w}(R), \lambda(R)Y, Q_{\tau(R)}(\lambda(R)Y \mid R)\right)\right], \tag{6}$$

where the adversarial reweighting effect is $a(\underline{w}, \bar{w}, \lambda y, q) \equiv (\bar{w} - \underline{w})\, 1\left\{\lambda y > q\right\} - (1 - \underline{w})$. The expression for the lower bound is analogous.

Theorem 1 tells us that the solution can be written in closed form, and the upper bound on the object of interest can be written simply as a moment. The infimum has a similar solution. These results allow estimation to be simple and fast. The solution is also highly intuitive: we have a quantile balancing object and a corresponding $\tau(R)$ to ensure that $E[W|R] = 1$ is satisfied. In the supremum problem, once $\lambda(R)Y$ is above a quantile threshold, we want to place the largest weights possible on those observations, and for $\lambda(R)Y$ below that threshold, we place the lowest weights possible. The expressions for $a(.)$ are also immediate from the W^* solutions. In the supremum problem, when $\lambda y > q$, $a(.) = \bar{w} - 1$; when $\lambda y < q$, $a(.) = \underline{w} - 1$. We do the opposite for the infimum.

The proof proceeds by using a less constrained quantile balancing problem than (3) that is easier to solve. Once we have shown that W^* solves the less constrained problem, and it is feasible in (3), then the W^* must also be the solution to (3). Details are in Appendix A.2.

There may be situations where researchers are not willing to bound the likelihood ratio above. Since W corresponds to a ratio of densities, this one-sided phenomenon occurs when the support of P_{Obs} is strictly contained in the support of P_{True}. In particular, we may be interested in upper bounds on the object of interest that take the following form:

$$\sup_{W} E\left[W\lambda(R)Y\right] s.t. W \in \left[\underline{w}\left(R\right), \infty\right), E[W|R] = 1 \tag{7}$$

The characterization in Theorem 1 is insufficient. For example, if $\bar{w}(R)$ is infinite and $\lambda(R)Y$ is always positive, then $\tau(R) = 1$ and the formula from Equation (6) is $E[\lambda(R)Y\underline{w}(R)] < E[\lambda(R)Y]$. Is there a way to nest the identification results in the finite-$\bar{w}$ case and the infinite-$\bar{w}$ case?

We show that there is a convenient characterization of sharp bounds that holds even if the likelihood ratio is unbounded. This characterization also slightly generalizes the environment of Theorem 1 to hold even when Y contains mass points.

To motivate the generalization, observe that it is innocuous to subtract $E[Q_\tau a(\cdot)] = 0$ from the Equation (6) because $E[a(\cdot) \mid R] = 0$. While it is unnecessary when $\bar{w}$ is finite and Y is continuously distributed, this modified characterization is useful when we allow $\bar{w}$ to be infinite.

Theorem 2. *Whether $\bar{w}(R)$ is finite or infinite, the upper bound of (3) is given by*

$$E\left[\lambda(R)Y + \left(\lambda(R)Y - Q_{\tau(R)}\left(\lambda(R)Y|R\right)\right) a(\underline{w}(R), \bar{w}(R), \lambda(R)Y, Q_{\tau(R)}(\lambda(R)Y \mid R))\right]. \tag{8}$$

and an analogous expression holds for the lower bound.

Equation (8) is a slight but useful generalization of Equation (6). When the assumptions of Theorem 1 hold, $E[a(\cdot) \mid R] = 0$ and the expressions will be identical. Further, the solution W^* will be identical in both problems. Theorem 2 is more general in that, by subtracting Q_τ, we can now accommodate unbounded outcomes. When $\lambda(R)Y$ has a point mass at $Q_{\tau(R)}(\cdot)$, the adversarial reweighting effect a is not conditionally mean-zero and the the characterization (6) is not valid. On that event where a is incorrectly defined, the generalized characterization (8) multiplies a by zero and so remains the sharp bound. A similar characterization is used in the supplement of Dorn and Guo (2022) in considerably less generality.

To intuit the potentially finite bounds when the likelihood ratio bound $\bar{w}$ is infinite, the adversarial distribution can put unbounded weight on large outcomes, but it must put at least a likelihood ratio of $\underline{w}(R)$ on observed outcomes. If $\bar{w}(R)$ is infinite, then $\tau(R) = 1$ and we never observe $\lambda(R)Y > Q_1$. Instead, the term $(\lambda(R)Y - Q_1)a$ is always equal to $(Q_1 - \lambda(R)Y)(1 - \underline{w}(R))$. Hence, if $\underline{w}(R) < 1$ and Q_1 is infinite, then the upper bound is always infinite. If $\underline{w}(R) = 1$ but $\bar{w}(R)$ and Q_1 are infinite, then the upper bound is defined awkwardly but the claim holds under the notation $-\infty * 0 = 0$.

3 Applications

We provide new identification results using our framework for three settings: regression discontinuity with one-sided manipulation, inverse propensity weighting with treatment selected on unobservables, and ordinarily least squares with confounding. Our analysis is immediate after appropriate definition of variables. Our measure can naturally be interpreted as a selection model in the RD and IPW cases.

3.1 Sharp Regression Discontinuity

We substantially generalize the state of the art sensitivity analysis for sharp RD using our framework and immediately obtain bounds on more standard causal estimands.

We are interested in treatment effects $Y(1)-Y(0)$, but we only observe $Y = DY(1)+(1-D)Y(0)$. We assume that observations are drawn independently from a full distribution over $(M, X(1), X(0), Y(1), Y(0), D)$, where M is a binary manipulation variable $X(m)$ is the potential running variable when $M = m$, $D \in \{0, 1\}$ is the treatment status, and $Y(d)$ is the potential outcome when $D = d$. Observations who are manipulators (i.e., $M = 1$) endogenously manipulate their X to sort into treatment. However, we face the fundamental problem of causal inference and only observe independent tuples of the coarsening random variables (X, Y, D). $X = X(M)$ is the running variable, $Y = Y(D)$ the outcome, and D the treatment. We use the notation "$X \approx c$" to mean that we take the limit as $\varepsilon \to 0$ of $X \in [c-\varepsilon, c+\varepsilon]$. Further, $X = c^+$ denotes $X \in [c, c+\varepsilon]$ and $X = c^-$ denotes $X \in [c-\varepsilon, c]$.

We focus on sharp RD designs. The main assumption in sharp RD is the following:

Assumption 1 (Sharp RD)**.** $P(D = 1|X > c) = 1, P(D = 1|X < c) = 0$

Assumption 1 is that we study a sharp RD setup: treatment D is assigned for all observations on one side of a cutoff c but not on the other. It mimics assumption (RD) of Hahn et al. (2001). It is observationally testable and often obvious in applications: if X is the net vote share of an election candidate, then the candidate wins if and only if $X > 0$.

We assume that every observation can be partitioned into manipulators and non-manipulators. Non-manipulator ($M = 0$) running variables are as good as random around the discontinuity.

Assumption 2 (Non-manipulator exogeneity)**.** $(Y(1), Y(0)) \perp\!\!\!\perp X \mid M = 0, X \approx c$. $F_{X|M=0}(x)$ is differentiable in x at c with a positive derivative.

In the standard RD setup, Assumption 2 is equivalent to assuming that the conditional independence assumption holds for all observations. In our more general context, this conditional independence is only required for the non-manipulators — no such assumption is imposed on the manipulators so far. Differentiability and exogeneity implies that the change in non-manipulator average outcomes across the cutoff c is the non-manipulator average treatment effect. However, we cannot observe that average when $\tau > 0$ and there are manipulators.

We assume that manipulation occurs in one direction.

Assumption 3 (One-sided manipulation)**.** $F_{X|M=1}(c) = 0$. $F_{X|M=1}(x)$ is right-differentiable in x at c.

Assumption 3 requires manipulators to manipulate only in the direction of treatment. The notation that manipulators are treated is for convenience, since D and $1-D$ can be interchanged to remain consistent with our assumption. In the classic RD setup with no manipulation across the boundary at a sufficiently fine level, the conditional density $F_{X|M=1}$ is zero to the right of the cutoff. Settings where manipulation could plausibly occur in either direction is an interesting direction for future analysis.

The setup so far is identical to Gerard et al. (2020) and allows for one-sided manipulation. Using their notation, of the people just above the cutoff, the proportion who have manipulated is:

$$\tau := \mathbb{P}(M=1 \mid X=c^+)$$

τ is point identified. To avoid ambiguity with Gerard et al.'s notation, we use "$\tau(R)$" to refer to percentiles in our identification results applied here. Estimands that we may be interested in include the conditional average treatment effect (CATE), the conditional local average treatment effect (CLATE), and the conditional average treatment effect on the treated (CATT).

$$\begin{aligned}
\psi^{CATE} &:= \mathbb{E}\left[Y(1)-Y(0) \mid X \approx c\right] \\
\psi^{CLATE} &:= \mathbb{E}\left[Y(1)-Y(0) \mid X \approx c, M=0\right] \\
\psi^{CATT} &:= \mathbb{E}\left[Y(1)-Y(0) \mid X \approx c, X>c\right]
\end{aligned}$$

We call $\mathbb{E}\left[Y(1)-Y(0) \mid X \approx c, M=0\right]$ the CLATE because it is an average treatment effect at the cutoff among the population for whom the treatment is randomly assigned at the cutoff.

We write the causal estimands of interest in terms of conditional expectations of observed outcomes and potential outcomes as follows:

Lemma 1. *Under Assumptions 1-3, our main estimands of interest have the following expressions:*

$$\begin{aligned}
\psi_{CATE} &= \frac{1}{2-\tau}\mathbb{E}\left[Y \mid X=c^+\right] + \frac{1-\tau}{1-2\tau}\mathbb{E}\left[Y(1) \mid M=0, X \approx c\right] \\
&\quad - \frac{\tau}{2-\tau}\mathbb{E}\left[Y(0) \mid M=1, X \approx c\right] - \left(1-\frac{\tau}{2-\tau}\right)\mathbb{E}\left[Y \mid X=c^-\right] \\
\psi_{CATT} &= \frac{\mathbb{E}\left[(2D-1)Y \mid X \approx c\right] - \frac{\tau}{2-\tau}\mathbb{E}_{\mathbb{P}}[Y(0) \mid X \approx c, M=1]}{\frac{\tau}{2-\tau} + \left(1-\frac{\tau}{2-\tau}\right)/2} \\
\psi_{CLATE} &= \mathbb{E}_{\mathbb{P}}[Y(1) \mid X \approx c, M=0] - \mathbb{E}[Y \mid X=c^-]
\end{aligned}$$

Most of the quantities in the expressions above are observable from the data. The only unobservable objects are $\mathbb{E}\left[Y(1) \mid M=0, X \approx c\right]$ and $\mathbb{E}\left[Y(0) \mid M=1, X \approx c\right]$. We will show how these objects can be written as observable expectations of outcomes weighted by unobserved likelihood ratios and how our procedure can place bounds on them.

Lemma 2 shows that the relevant distributions can be written in terms of confounded probabilities of manipulation.

Lemma 2. *Suppose $\mathbb{Q}$ is a distribution over $(Y(1), Y(0), M, D(1), D(0))$ that satisfies the GRR assumptions and define $\mathbb{Q}$'s manipulation selection functions $q_1(y_1) \equiv \mathbb{Q}(M=1|Y(1)=y_1)$ and $q_0(y_0) \equiv \mathbb{Q}(M=1|Y(0)=y_0)$. Then these translate the observed outcome likelihoods to unobserved likelihoods as follows:*

$$d\mathbb{Q}(Y(1) \mid X \approx c, M=0) = \frac{1}{1-\tau}\frac{1-q_1(Y(1))}{1+q_1(Y(1))}d\mathbb{P}(Y(1) \mid X=c^+)$$

$$d\mathbb{Q}(Y(0) \mid X \approx c, M = 1) = \frac{2(1-\tau)}{\tau} \frac{q_0(Y(0))}{1 - q_0(Y(0))} d\mathbb{P}(Y(0) \mid X = c^-)$$

Due to Lemma 2,

$$E\left[Y(1)|X \approx c, M = 0\right] = \frac{1}{1-\tau} E\left[Y \frac{1-q_1}{1+q_1} \frac{1\left\{X = c^+\right\}}{P\left(X = c^+\right)}\right] \tag{9}$$

$$E\left[Y(0)|X \approx c, M = 1\right] = \frac{2\left(1-\tau\right)}{\tau} E\left[Y \frac{q_0}{1-q_0} \frac{1\left\{X = c^-\right\}}{P\left(X = c^-\right)}\right] \tag{10}$$

To place bounds on CLATE, we simply have to place bounds on $E\left[Y(1)|X \approx c, M = 0\right]$, which has the form above that is amenable to using Theorem 1. In particular,

Proposition 1. *Suppose Assumptions 1-3 hold and the sensitivity assumption is*

$$\frac{\mathbb{P}(M = 1 \mid Y(1), X \approx c)}{\mathbb{P}(M = 0 \mid Y(1), X \approx c)} \Big/ \frac{\tau}{2(1-\tau)} \in [\Lambda_1^-, \Lambda_1^+].$$

Then the upper bound for $E\left[Y(1)|X \approx c, M = 0\right]$ is identical to solving (3) with

$$\begin{aligned} \lambda_{True}(R) &= \frac{1\left\{X = c^+\right\}}{P\left(X = c^+\right)} \frac{1-q_1}{1+q_1} \frac{1}{1-\tau} \\ \lambda(R) &= \frac{1\left\{X = c^+\right\}}{P\left(X = c^+\right)}, \qquad W = \frac{1-q_1}{1+q_1} \frac{1}{1-\tau} \\ \underline{w}(R) &= \frac{1}{1-\tau+\tau\Lambda_1^+}, \qquad \bar{w}(R) = \frac{1}{1-\tau+\tau\Lambda_1^-} \end{aligned}$$

The factor of two is necessary to write the restriction in terms of τ and a parameter which corresponds to exogeneity at $\Lambda^- = \Lambda^+ = 1$. Gerard et al. (2020) placed bounds on ψ^{CLATE} but not ψ^{CATE} when $\Lambda^+ = \infty$ and $\Lambda^- = 0$.

Similarly, to place bounds on CATT, we simply place bounds on $E\left[Y(0)|X \approx c, M = 1\right]$,

Proposition 2. *Suppose Assumptions 1-3 hold and the sensitivity assumption is*

$$\frac{\mathbb{P}(M = 1 \mid Y(0), X \approx c)}{\mathbb{P}(M = 0 \mid Y(0), X \approx c)} \Big/ \frac{\tau}{2(1-\tau)} \in [\Lambda_0^-, \Lambda_0^+].$$

Then the upper bound for $E\left[Y(0)|X \approx c, M = 1\right]$ is identical to solving (3) with

$$\begin{aligned} \lambda_{True}(R) &= \frac{q_0}{1-q_0} \frac{1\left\{X = c^-\right\}}{P\left(X = c^-\right)} \frac{2\left(1-\tau\right)}{\tau} \\ \lambda(R) &= \frac{1\left\{X = c^-\right\}}{P\left(X = c^-\right)}, \qquad W = \frac{q_0}{1-q_0} \frac{2\left(1-\tau\right)}{\tau} \\ \underline{w}(R) &= \Lambda_0^-, \qquad \bar{w}(R) = \Lambda_0^+ \end{aligned}$$

For the CATE, we want to place bounds on

$$\psi = \frac{1-\tau}{1-2\tau} E\left[Y(1)|M = 0, X \approx c\right] - \frac{\tau}{2-\tau} E\left[Y(0)|M = 1, X \approx c\right] \tag{11}$$

Proposition 3. *Suppose the assumptions from Proposition 1 and 2 hold. Finding the upper bound for ψ is* *identical to solving* (3) *with*

$$\lambda_{True}(R) = \frac{1}{1-2\tau}\frac{1-q_1}{1+q_1}\frac{1\{X=c^+\}}{P(X=c^+)} - \frac{2(1-\tau)}{2-\tau}\frac{q_0}{1-q_0}\frac{1\{X=c^-\}}{P(X=c^-)}$$

$$\lambda(R) = \frac{1\{X=c^+\}(2-\tau)(1-\tau)P(X=c^-) - \tau 1\{X=c^-\}(1-2\tau)P(X=c^+)}{(1-2\tau)(2-\tau)P(X=c^+)P(X=c^-)}$$

$$W = \frac{1-q_1}{1+q_1}1\left\{X=c^+\right\}\frac{1}{(1-\tau)} + 2(1-\tau)\frac{q_0}{1-q_0}1\left\{X=c^-\right\}\frac{1}{\tau}$$

$$\underline{w}(R) = 1\left\{X=c^+\right\}\frac{1}{1-\tau+\tau\Lambda_1^+} + 1\left\{X=c^-\right\}\Lambda_0^-, \qquad \bar{w}(R) = 1\left\{X=c^+\right\}\frac{1}{1-\tau+\tau\Lambda_1^-} + 1\left\{X=c^-\right\}\Lambda_0^+.$$

As before, sharp bounds can be obtained in closed form by applying Theorem 1. Our approach tightens bounds from Gerard et al. (2020), and allows meaningful bounds to be placed on CATT and CATE that were previously unachievable.

3.2 Inverse Propensity Weighting

We substantially generalize existing IPW identification results using our framework.

We are interested in expected potential outcomes or treatment effects of a binary treatment Z with observed controls X; the observable quantities are $R = (X, Z)$. We assume that there is a full distribution P over covariates, potential outcomes $Y(1)$ and $Y(0)$, and treatment Z generating the tuple $(X, Y(1), Y(0), Z, U)$. Unconfoundedness conditional on potential outcomes is trivial: $Y(1) \perp\!\!\!\perp Z \mid X, Y(1)$. However, we only observe the coarsened distribution over (X, Y, Z) where $Y = Y(Z)$.

We measure failures of unconfoundedness in terms of treatment selection. Unconfoundedness is the assumption that $Y(0), Y(1) \perp Z|X$. Define the propensity for treatment given controls as $e(X) = P(Z = 1 \mid X)$ and the unobserved propensity for treatment given controls and potential outcome as $e_z(x, y) = P(Z = 1 \mid x, Y(z) = y)$. If unconfoundedness holds,

$$\frac{(e_z(x,y))/(1-e_z(x,y))}{e(X)/(1-e(X))} = 1.$$

If unconfoundedness fails, then $\frac{(e_z(x,y))/(1-e_z(x,y))}{e(X)/(1-e(X))}$ is not equal to one.

We derive meaningful bounds on causal objects of interest under limited violations of unconfoundedness. In particular, we consider functions $l_0(X)$ and $u_0(X)$ that bound the odds ratio shift from observing the potential outcomes. Formally, we assume that $e_z(X, Y(z)) \in (0, 1)$ almost surely for both treatment assignments z and:

$$l_z(X) \leq \frac{e_z(X,Y(z))/(1-e_z(X,Y(z)))}{e(X)/(1-e(X))} \leq u_z(X). \tag{12}$$

It is immediately clear that Equation (12) can equivalently be viewed in terms of how far e_z and e can differ, as proposed by Masten and Poirier (2018). However, this odds ratio-based parameterization adapted from Tan (2006) (where $l_z = \Lambda^{-1}$ and $u_z = \Lambda$ uniformly) can be viewed as a bound on the likelihood ratio between unobserved and observed potential outcomes and as a result is more convenient for empirical analysis.

We begin in the simpler case of bounding $E[Y(1)]$. If $e_0(X, U) > 0$ almost surely, then $E[Y(1)] = E[YZ/e_1(X, Y(1))]$. Hence, bounds can be placed by using Theorem 1 applied to the objects defined in the proposition below.

Proposition 4. *Finding the upper bound for $E[Y(1)]$ under Equation (12) is identical to solving (3) with*

$$\lambda_{True}(R) = \frac{Z}{e_1(Z, Y(1))}, \qquad \lambda(R) = \frac{Z}{e(X)}, \qquad W = \frac{e(X)}{e_1(Z, Y(1))}$$
$$\underline{w}(R) = (1 - e(X))\frac{1}{u_1(X)} + e(X), \qquad \bar{w}(R) = (1 - e(X))\frac{1}{l_1(X)} + e(X)$$

The result is intuitive. The w bounds would be 1 when $l_1 = u_1 = 1$. As $e(X)$ gets closer to one, we see a greater share of $Y(1)$ and the w bounds get closer to one since we are interested in the $Y(1)$ only. We can add the other $1 - e(X)$ to varying degrees to fit the sensitivity assumption. An analogous approach can be used to obtain $E[Y(0)] = E[Y(1 - Z)/(1 - e_0(X, Y(0)))]$. The result in Proposition 4 extends the analysis of Tan (2022); Frauen et al. (2023) to allow e_1 arbitrarily small.

Researchers may also be interested in the average treatment effect (ATE) $E[Y(1) - Y(0)]$. There is no restriction in the model that $e_1 = e_0$, but we require both unobserved propensities to satisfy (12). We are allowed to use different latent objects for Z and $1 - Z$ because we do not ask for the same unobserved confounder U for different partitions of the data. Namely, we can use $e_1 = P(Z = 1|X, Y(1))$ and $e_0 = P(Z = 1|X, Y(0))$, and the sensitivity assumption holds for both e_z. In the respective problems, it is integrating over the respective potential outcomes that matters. Our analysis proceeds similarly.

Proposition 5. *For (12), finding the upper bound for $E[Y(1) - Y(0)]$ is identical to solving (3) with*

$$\lambda_{True}(R) = \frac{Z}{e_1(X, Y(1))} - \frac{1 - Z}{1 - e_0(X, Y(0))}, \qquad \lambda(R) = \frac{Z}{e(X)} - \frac{1 - Z}{1 - e(X)}, \qquad W = \frac{Ze(X)}{e_1(Z, Y(1))} + \frac{(1 - Z)(1 - e(X))}{1 - e_0(X, Y(0))}$$
$$\underline{w}(R) = Z\left(e(X) + \frac{1 - e(X)}{u_1}\right) + (1 - Z)\left(1 - e(X) + e(X)l_0\right)$$
$$\bar{w}(R) = Z\left(e(X) + \frac{1 - e(X)}{l_1}\right) + (1 - Z)\left(1 - e(X) + e(X)u_0\right)$$

The c-dependence sensitivity assumption of Masten and Poirier (2018) can be expressed as $e_z(X, Y(z)) \in [e(X) - c, e(X) + c] \cap [0, 1]$. Then, using our existing notation, if $e(X) \in (c, 1 - c)$, we have:

$$u_z = \frac{(e(X) + c)/(1 - (e(X) + c))}{e(X)/(1 - e(X))}$$
$$l_z = \frac{(e(X) - c)/(1 - (e(X) - c))}{e(X)/(1 - e(X))}.$$

Then, bounds on the ATE can be obtained analogously using our framework.

3.3 Ordinary Least Squares

We illustrate novel bounds for Ordinary Least Squares using our framework. We are interested in a linear combination of coefficients β from a hypothetical linear model:

$$Y = X\beta + u.$$

We are interested in $\delta'\beta$, where X includes an intercept but δ puts no weight on the intercept term. We believe without loss of generality that $E[u] = 0$. However, there is confouding in the sense that $E[X\varepsilon] \neq 0$. If we re-weighted the data by $W = \frac{dP(u)}{dP(u|X)}$, we would obtain $E[W(Y - X\beta) \mid X] = E[uW \mid X] = 0$ and the reweighted least squares would obtain the correct coefficients.

The sensitivity assumption is then on $W = \frac{dP_{True}}{dP_{Obs}}$. For instance, we may have:

$$\underline{w} \leq W = \frac{dP_{True}}{dP_{Obs}} \leq \bar{w}$$

Using the notation of our general framework,

$$\lambda(R) = \delta' E[X'X]^{-1} X$$
$$\lambda_{True}(R) = \delta' E[X'X]^{-1} X \frac{dP_{True}}{dP_{Obs}}$$

Then, Theorem 1 or Theorem 2 can be applied to obtain bounds on the object of interest.

This result has implications on any OLS-based design, including event studies and difference in differences (DD). The identifying assumption in DD and event studies is implied by the parallel trends assumption, and there may be concern that parallel trends need not be empirically credible.

There are existing ways to do sensitivity analysis to the failure of parallel trends. For example, Rambachan and Roth (2023) places an explicit assumption on the extent that the slopes are not parallel. The approach we present here is calibrated in terms of likelihood ratios rather than model coefficients. As a result, our unobserved confounding measure is invariant to taking transformations like logarithms of outcomes but may be less interpertable for practitioners.

4 Implementation

We illustrate the procedure using a simulation in the c-dependence context of Section 3.2.

Our distribution of data is $X \sim U[-\eta, \eta]$, $Z \mid X \sim Bern(1/(1 + exp(-X)))$, and $Y \mid X, Z \sim \mathcal{N}((2 + X)(Z - 1), 1)$, where η is chosen so that the support of $e(X)$ is $[0.1, 0.9]$.

We consider $c = 0, 0.01, ..., 0.1$. When $c > 0.1$, the identified set is unbounded. The case $c = 0.1$ allows unbounded propensities, but we show in Corollary 1 (Page 15) that the identified set remains uniformly bounded for all $c \leq 0.1$.

We obtain bounds numerically. In particular, we average the closed-form identified set for $E[Y(1) - Y(0) \mid X]$ over one million draws of X. With the true ψ^+ and ψ^- essentially known, we can assess the validity of our estimation and inference procedures.

The bounds in the general program are estimated by plugging in the sample analog of Equation (8). We estimate λ, τ, $\underline{w}$, and $\bar{w}$ by plugging in propensity estimates $\hat{e}(X)$. The propensities are estimated using logistic regression of Z on X. We estimate the quantile function using quantile regression on 101 grid points ($\tau = 0, 0.01, ..., 1.00$). The quantile regression regresses $\hat{\lambda}Y$ on Z interacted with both X and $\hat{\lambda}(X)$. We estimate quantiles at 101 grid points ($\tau = 0, 0.01, ..., 1.00$). For each observation, we take the quantile regression corresponding to the closest grid point to the estimated $\hat{\tau}$.

Inference proceeds by a standard percentile bootstrap. For a given dataset, we can redraw observations with replacement. For every bootstrap draw b, we re-estimate propensities and weight bounds with propensities updated via one-step updating and estimate bootstrap upper and lower bounds $\hat{\psi}_b^+$ and $\hat{\psi}_b^-$. We do not re-estimate the quantile regression grid in bootstraps because, as we plan to show in the next version of this work, the quantile estimates have a second-order effect on the estimates. The 95% confidence interval for the identified set is the set bounded by the 2.5th quantile of the $\hat{\psi}_b^-$ draws and the 97.5th quantile of the $\hat{\psi}_b^+$ draws. The quantiles of the estimated bounds $\{\psi_b^-\}, \{\psi_b^+\}$ then form the confidence interval for the identified set. We also calculate two-sided 95% confidence intervals for the lower and upper bound analogously.

For a given sensitivity parameter c, we run 1,000 simulations of the data with 2,000 observations. Within each simulation, we take 1,000 bootstrap draws and calculate the bounds for each bootstrap draw.

We present our mean and median bound estimates in Figure 1. Our median bound estimates generally track the true bounds. Our mean bound estimates roughly track the identified set until c gets close enough to 0.1 that some simulations produce infinite estimated bounds (0.5% of simulations at $c = 0.07$). As c gets close to 0.1 and the most extreme $\tau(R)$ values get close to one, our median estimates become slightly too wide. We expect to show in future work that this reflects robustness of our characterization with respect to quantile errors, which are especially likely when applying our discrete grid to extreme $\bar{w}$ values.

Our coverage results are reported in Table 1. Our coverage rates are generally within the 95% exact coverage interval of 93.6% to 96.3%. As c gets higher, the bounds become slightly more conservative but still remain valid. As c gets close to 0.1, there is an increasing chance of estimating at least one observation's propensity $\hat{e}(X_i) \notin (c, 1-c)$ and potentially generating an infinite estimated bound. When $c = 0.1$, the identified set rests on a knife edge between $[1.5, 2.5]$ and $(-\infty, \infty)$. We find that in this case the confidence intervals cover the true (finite) identified set in 98.7% of simulations and are unbounded in 94.5% of simulations. (20.6% of bound estimates are unbounded at $c = 0.10$.)

5 Conclusion

This paper has proposed a novel sensitivity analysis framework for linear estimand identification failures. By placing bounds on the density ratio between the observed and true conditional outcome distributions, we obtain sharp and tractable analytic bounds. This framework generalizes existing sensitivity models in RD and IPW, generates a new sensitivity analysis for OLS, and provides new results for unbounded likelihood ratios. As a result of our general setting, we now have a procedure for sensitivity analysis of the CATE in RD that has previously remained an open issue; we have a simpler method of deriving bounds under c-dependence of Masten and Poirier (2018) in IPW; and we have a novel sensitivity framework in OLS that is based on likelihood ratios rather than functional form-dependent relationships.

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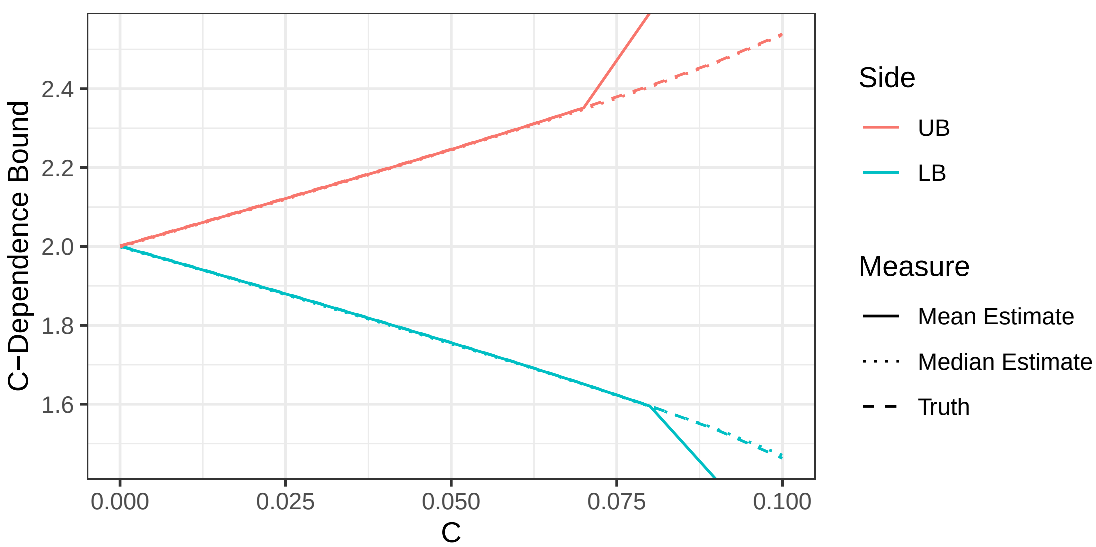

Figure 1: Average bound estimates (solid line), median bound estimates (dotted line), and true bounds (dashed line) in our 1,000 simulations. Our estimates are generally close to median unbiased. Once $c \geq 0.08$, some simulations estimate $\hat{e}(x) \notin (0.08, 0.92)$ and have infinite estimated identified sets.

Table 1: Coverage for 95% confidence set

C	CI Coverage (Target 95%)		
	Set	LB	UB
0.00	94.1	95.4	93.5
0.01	93.9	95.7	93.7
0.02	94.0	95.8	93.9
0.03	94.4	95.8	94.0
0.04	94.6	96.0	93.7
0.05	94.9	96.2	94.1
0.06	95.6	96.2	94.6
0.07	96.5	96.5	95.5
0.08	96.9	96.6	97.4
0.09	98.0	97.3	98.3
0.10	98.3	97.6	99.1

CI coverage denotes the percentage of simulations in which the 95% identified set confidence interval includes the true identified set and in which one-sided 95% lower and upper bound confidence intervals contain the true bounds.

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A Proofs

A.1 Additional Claims

The following claims show that the c-dependence identified set in our implementation example is finite and bounded for all $c \leq 0.1$ but is infinite for all $c > 0.1$.

Proposition 6. Bounds on identified set. *Consider the general identification setting and suppose $Y \mid R \sim \mathcal{N}(\mu(R), \sigma(R)^2)$. Then for all $\epsilon \in (0,1)$, the identified set is a subset of*

$$\left[E[\lambda(R)Y] \pm E\left[\sigma(R)\lambda(R)(1-\underline{w}(R))\left(\sqrt{2\log(\bar{w}(R))} + \sqrt{2/\pi} + (1-\underline{w}(R))^{\epsilon}\sqrt{1/(e*\epsilon)}\right)\right]\right],$$

where $[a \pm b]$ denotes the closed interval $[a-b, a+b]$.

Corollary 1. *Suppose $Y \mid X, Z \sim \mathcal{N}(\mu(X,Z), \sigma(X,Z)^2)$, the support of the observed propensity function $e(X)$ is the closed interval $[\eta_1, 1-\eta_2] \subset (0,1)$, and the conditional outcome variance $\sigma(X,Z)$ is positive and bounded. Then there is a finite $B > 0$ such that the ATE identified set is a subset of $[E[\mu(X,1)-\mu(X,0)] - B, E[\mu(X,1)-\mu(X,0)] + B]$ for all $c < \min\{\eta_1,\eta_2\}$ but is $(-\infty,\infty)$ for all $c > \min\{\eta_1,\eta_2\}$.*

A.2 Proofs for Section 2

Proof of Theorem 1. A related problem is

$$\sup_{W} E[W\lambda(R)Y] \text{ s.t. } W \in [\underline{w}(R), \bar{w}(R)] \text{ and } E[WQ_{\tau(R)}(\lambda Y \mid R)] = E[Q_{\tau(R)}(\lambda Y \mid R)] \tag{13}$$

First observe that (3) is a more constrained problem than (13). To see this, when $E[W|R] = 1$,

$$E[WQ_{\tau(R)}(\lambda(R)Y \mid R)] = E\left[E\left[W|R\right]Q_{\tau(R)}(\lambda(R)Y \mid R)\right] = E[Q_{\tau(R)}(\lambda(R)Y \mid R)]$$

This means that if we can find a solution to (13) that is feasible in (3), then we have also found the solution to (3). To save on notation, since we are solving the sup problem, let $W^* = W^*_{sup}$. Since $E\left[W|R\right] = 1$,

$$\begin{aligned}
E\left[W^*|R\right] &= F\left(Q_{\tau(R)}\left(\lambda(R)Y|R\right)\right)E\left[\underline{w}(R)|R, y \leq Q_{\tau(R)}\left(\lambda(R)Y|R\right)\right] \\
&\quad + \left(1 - F\left(Q_{\tau(R)}\left(\lambda(R)Y|R\right)\right)\right)E\left[\bar{w}(R)|R, y > Q_{\tau(R)}\left(\lambda(R)Y|R\right)\right] \\
&= \tau(R)\underline{w}(R) + (1-\tau(R))\,\bar{w}(R) = 1
\end{aligned}$$

Then, we will have

$$\tau(R) = \frac{1-\bar{w}(R)}{\underline{w}(R)-\bar{w}(R)} = \frac{\bar{w}(R)-1}{\bar{w}(R)-\underline{w}(R)}$$

$\tau(R)$ is guaranteed to be between 0 and 1, because $\underline{w}(R) \leq 1 \leq \bar{w}(R)$. The proposed solution W^* by construction satisfies $E\left[W^*|R\right] = 1$, so it's feasible in (3). We can show that it is the solution to (13). First observe that, under (3), $E[W|R] = 1$, so

$$\begin{aligned}
E[W\lambda(R)Y] &= E\left[Q_{\tau(R)}(\lambda(R)Y|R)W + \left(\lambda(R)Y - Q_{\tau(R)}(\lambda(R)Y|R)\right)W\right] \\
&\leq E\left[Q_{\tau(R)}(\lambda(R)Y|R)\right] + E\left[\left(\lambda(R)Y - Q_{\tau(R)}(\lambda(R)Y|R)\right)W^*\right] \\
&= E\left[Q_{\tau(R)}(\lambda(R)Y|R)W^*\right] + E\left[\left(\lambda(R)Y - Q_{\tau(R)}(\lambda(R)Y|R)\right)W^*\right]
\end{aligned}$$

$$= E\left[\lambda(R)YW^*\right]$$

The inequality occurs because W took the maximum allowable value when $\lambda(R)Y > Q_{\tau(R)}(\lambda(R)Y|R)$ and the minimum allowable value when $\lambda(R)Y < Q_{\tau(R)}(\lambda(R)Y|R)$. Hence, W^* solves (13). Consequently, it solves (3).

The derivation for the lower bound is analogous. □

Proof of Theorem 2. First, we show the claim when $\bar{w}(R)$ is finite. With a mass point at $\lambda(R)Y = Q_{\tau(R)}(\lambda(R)Y \mid R)$, W^*_{sup} no longer satisfies $E[W^*_{sup} \mid R] = 1$. We thus construct a different solution $\tilde{W}^*_{sup}$ and its corresponding adversarial weighting $\bar{a}(.)$ that is valid. We will then show that using $\bar{a}(.)$ in the expression for the upper bound is equal to the analogous expression that simply uses $a(.)$.

We define a function $\bar{a}$ that is equal to a when $\lambda(R)Y \neq Q_{\tau(R)}(\lambda(R)Y \mid R)$ and is conditionally mean-zero even if $\lambda(R)Y$ contains mass points. We will use a function $\alpha(r)$ as an input to construct appropriate weight when $\lambda(R)Y = Q_{\tau(R)}(\cdot)$.

There is a real-valued function $\alpha(r)$ satisfying $\alpha(R) \in [0,1]$ almost surely and:

$$\begin{aligned}
&(\bar{w}(R) - \underline{w}(R))\mathbb{P}(\lambda(R)Y = Q_{\tau(R)}(\lambda Y|R) \mid R)\ \alpha(R) \\
=&\underline{w}(R)\mathbb{P}(\lambda(R)Y < Q_{\tau(R)}(\lambda Y|R) \mid R) + \bar{w}(R)\mathbb{P}(\lambda(R)Y \geq Q_{\tau(R)}(\lambda Y|R) \mid R) - 1 \text{ a.s.}
\end{aligned}$$

When $\bar{w} = \underline{w}$ or $\mathbb{P}(\lambda(R)Y = Q_{\tau(R)}(\lambda Y|R)) = 0$, the construction holds for any $\alpha(R) \in [0,1]$. When $\bar{w} > \underline{w}$, the claim $\alpha(R) \geq 0$ holds because the right-hand side of the equality is at least $\underline{w}(R)\tau(R) - 1 + \bar{w}(R)(1 - \tau(R)) = 0$ and the claim $\alpha(R) \leq 1$ holds by algebra that we omit for brevity.

$\alpha(R)$ has the following convenient property:

$$\begin{aligned}
&\underline{w}(R)\mathbb{P}(\lambda(R)Y < Q_{\tau(R)}(\lambda Y|R) \mid R) + \bar{w}(R)\mathbb{P}(\lambda(R)Y > Q_{\tau(R)}(\lambda Y|R) \mid R) \\
&+(\alpha(R)\underline{w}(R) + (1-\alpha(R))\bar{w}(R))\mathbb{P}(\lambda(R)Y = Q_{\tau(R)}(\lambda Y|R) \mid R) \\
=&\underline{w}(R)\mathbb{P}(\lambda(R)Y < Q_{\tau(R)}(\lambda Y|R) \mid R) + \bar{w}(R)\mathbb{P}(\lambda(R)Y \geq Q_{\tau(R)}(\lambda Y|R) \mid R) \\
&+\alpha(R)(\underline{w}(R) - \bar{w}(R))\mathbb{P}(\lambda(R)Y = Q_{\tau(R)}(\lambda Y|R) \mid R) \\
=&1 \text{ a.s.}
\end{aligned}$$

We define $\bar{a}(.)$ in the following way:

$$\bar{a}(\underline{w}, \bar{w}, \lambda y, q; r) \equiv \begin{cases} \alpha(r)\underline{w}(r) + (1-\alpha(r))\bar{w}(r) - 1 & \text{if } r \in supp(R), \lambda y = q = Q_{\tau(r)} \\ a(\underline{w}, \bar{w}, \lambda y, q) & \text{otherwise} \end{cases} \tag{14}$$

The expression of $\bar{a}$ is the consequence of using

$$\tilde{W}^*_{sup} = \begin{cases} \bar{w}(R) & if \quad \lambda(R)Y > Q_{\tau(R)}\left(\lambda(R)Y|R\right) \\ \underline{w}(R) & if \quad \lambda(R)Y \leq Q_{\tau(R)}\left(\lambda(R)Y|R\right) \\ \alpha(R)\underline{w}(R) + (1-\alpha(R))\bar{w}(R) & if \quad \lambda(R)Y = Q_{\tau(R)}\left(\lambda(R)Y|R\right) \end{cases}, \tau(R) = \frac{\bar{w}(R) - 1}{\bar{w}(R) - \underline{w}(R)}$$

as the solution to the supremum problem. In the more general setup that permits point masses, $\alpha(R)$ is constructed so that $E[\tilde{W}^*_{sup}|R] = 1$ and $E[\bar{a}(\cdot) \mid R] = 0$. Hence, $E[\lambda(R)Y + \lambda(R)Y\bar{a}(\cdot)]$ is a sharp bound when $\bar{w}$ is finite by an adaptation of the proof of Theorem 1.

When $\bar{w}$ is finite, the sharp bound $E[\lambda(R)Y + \lambda(R)Y\bar{a}(\cdot)]$ is equal to $E[\lambda(R)Y + (\lambda(R)Y - Q_{\tau(R)}(\lambda(R)Y \mid R)\bar{a}(\cdot)]$ because $E[\bar{a}(\cdot) \mid R] = 0$ almost surely. Changing $\bar{a}$ to a when $\lambda(R)Y = Q_{\tau(R)}(\cdot)$ has no effect on this characterization: $E[\lambda(R)Y + (\lambda(R)Y - Q_{\tau(R)}(\lambda(R)Y \mid R)\bar{a}(\cdot)] = E[\lambda(R)Y + (\lambda(R)Y - Q_{\tau(R)}(\lambda(R)Y \mid R)a(\cdot)] =$ (8).

It only remains to verify the claim when $\bar{w}(R)$ is infinite. For simplicity, we verify the claim when $\bar{w}(R)$ is infinite almost surely. When $\bar{w}(R) = \infty$, we have $\tau(R) = 1$. Hence,

$$
\begin{aligned}
&\sup_W E\left[W\lambda(R)Y\right] s.t. W \in [\underline{w}\left(R\right), \infty), E[W|R] = 1 \\
=&\sup_W E\left[\underline{w}(R)\lambda(R)Y + (W - \underline{w}(R))\lambda(R)Y\right] s.t. W \in [\underline{w}\left(R\right), \infty), E[W - \underline{w}(R)|R] = 1 - \underline{w}(R) \\
=&E[\underline{w}(R)\lambda(R)Y + (1 - \underline{w}(R))Q_1(\lambda(R)Y \mid R)],
\end{aligned}
$$

where the second equality follows by putting all weight at the supremum of the support of $\lambda(R)Y \mid R$. The proposed bound (8) is equal to:

$$
E\left[\underline{w}(R)\lambda(R)Y + (1 - \underline{w}(R))Q_1\left(\lambda(R)Y|R\right)\right].
$$

because we never observe $\lambda(R)Y > Q_1$, so the term $(\lambda(R)Y - Q_1)a$ is always equal to $(Q_1 - \lambda(R)Y)(1 - \underline{w}(R))$. The two are equal, completing the proof. □

A.3 Proofs for Section 3

To prove the propositions, we first prove a few useful lemmas.

Lemma 3. *Suppose $F_{X|M=0}(x)$ is differentiable in x at c with a positive derivative. Then $\mathbb{P}(X > c \mid X \approx c, M = 0) = 1/2$.*

Proof of Lemma 3. Define $f_{x|M=0}(c) > 0$ to be the derivative of $F_{X|M=0}(x)$ at c. Then we have:

$$
\begin{aligned}
\mathbb{P}(X > c \mid X \approx c, M = 0) &= \lim_{\varepsilon \to 0^+} \frac{F_{X|M=0}(c+\varepsilon) - F_{X|M=0}(c)}{F_{X|M=0}(c+\varepsilon) - F_{X|M=0}(c-\varepsilon)} \\
&= \lim_{\varepsilon \to 0^+} \frac{F_{X|M=0}(c+\varepsilon) - F_{X|M=0}(c)}{F_{X|M=0}(c+\varepsilon) - F_{X|M=0}(c) + F_{X|M=0}(c) - F_{X|M=0}(c-\varepsilon)} \\
&= \lim_{\varepsilon \to 0^+} \frac{\frac{F_{X|M=0}(c+\varepsilon) - F_{X|M=0}(c)}{\varepsilon}}{\frac{F_{X|M=0}(c+\varepsilon) - F_{X|M=0}(c)}{\varepsilon} + \frac{F_{X|M=0}(c) - F_{X|M=0}(c-\varepsilon)}{\varepsilon}} \\
&= \frac{\lim_{\varepsilon \to 0^+} \frac{F_{X|M=0}(c+\varepsilon) - F_{X|M=0}(c)}{\varepsilon}}{\lim_{\varepsilon \to 0^+} \frac{F_{X|M=0}(c+\varepsilon) - F_{X|M=0}(c)}{\varepsilon} + \lim_{\varepsilon \to 0^-} \frac{F_{X|M=0}(c+\varepsilon) - F_{X|M=0}(c)}{\varepsilon}} \\
&= \frac{f_{x|M=0}(c)}{2f_{x|M=0}(c)} = 1/2
\end{aligned}
$$

□

For our proofs, it is useful to use τ_0 instead. We define:

$$
\tau_0 := \mathbb{P}(M = 1 \mid X \approx c)
$$

Under the assumptions of the setup, there is a bijection between τ and τ_0. Since τ is point-identified by Gerard et al. (2020), τ_0 is also point identified.

Lemma 4. *Under Assumptions 1-3,* $\tau = 2\tau_0/(1+\tau_0)$ *and* $\tau_0 = \tau/(2-\tau)$.

Proof of Lemma 4.

$$\begin{aligned}\tau_0 := P(M=1|X \approx c) &= P(X=c^+|X \approx c)P(M=1|X=c^+) \\ &= P(X=c^+|X\approx c)\tau\end{aligned}$$

Due to Lemma 3,

$$\begin{aligned}P(X=c^+|X\approx c) &= P(M=1|X\approx c) + \frac{1}{2}P(M=0|X\approx c) \\ &= \tau_0 + (1-\tau_0)/2 = (1+\tau_0)/2\end{aligned}$$

Hence, $\tau_0 = ((1+\tau_0)/2)\tau$. Make τ and τ_0 the subject of formula respectively to obtain the result. □

Proof of Lemma 1. For the CATE, first observe that $\mathbb{P}(X < c \mid X \approx c) = (1-\tau_0)/2$. To see this, $\mathbb{P}(X < c|X \approx c) = \mathbb{P}(X < c|M=0, X\approx c)\mathbb{P}(M=0|X\approx c) = \mathbb{P}(X<c|M=0,X\approx c)(1-\tau_0) = (1-\tau_0)/2$, due to Lemma 3.

$$\begin{aligned}\psi_{CATE} &:= E[Y(1)-Y(0)|X\approx c] \\ &= E[Y(1)|X\geq c, X\approx c]P(X\geq c|X\approx c) + E[Y(1)|X<c, X\approx c]P(X<c|X\approx c) \\ &\quad - E[Y(0)|X\approx c, M=1]P(M=1|X\approx c) - P(M=0|X\approx c)E[Y(0)|X\approx c, M=0] \\ &= E[Y(1)|X\geq c, X\approx c]\frac{1}{2-\tau} + E[Y(1)|X<c, X\approx c]\frac{1-\tau_0}{2} \\ &\quad - E[Y(0)|X\approx c, M=1]\tau_0 - (1-\tau_0)E[Y(0)|X\approx c, M=0]\end{aligned}$$

The final equality uses the definition that $\tau_0 = P(M=1|X\approx c)$ and the result that $P(X<c|X\approx c) = (1-\tau_0)/2$.

Finally, $E[Y|X=c^+] = E[Y(1)|X=c^+]$ and $E[Y|X=c^-] = E[Y(0)|X=c^-]$ due to the sharp RD design of Assumption 1.

For the CATT, observe:

$$\begin{aligned}\psi_{CATT} &= \frac{\mathbb{E}[DY \mid X\approx c] - \mathbb{E}[DY(0)\mid X\approx c]}{\mathbb{E}[D\mid X\approx c]} \\ &= \frac{\mathbb{E}[DY\mid X\approx c]}{\mathbb{E}[D\mid X\approx c]} \\ &- \frac{\mathbb{P}[D=1, M=0\mid X\approx c]\mathbb{E}[Y\mid X=c^-]}{\mathbb{E}[D\mid X\approx c]} \\ &- \frac{\mathbb{P}[D=1, M=1\mid X\approx c]\mathbb{E}[Y\mid X\approx c, M=1]}{\mathbb{E}[D\mid X\approx c]} \\ &= \frac{\mathbb{E}[DY\mid X\approx c]}{\mathbb{E}[D\mid X\approx c]} \\ &- \frac{\mathbb{P}[D=0, M=0\mid X\approx c]\mathbb{E}[Y\mid X=c^-]}{\mathbb{E}[D\mid X\approx c]}\end{aligned}$$

$$- \frac{\tau_0 \mathbb{E}[Y \mid X \approx c, M = 1]}{\mathbb{E}[D \mid X \approx c]}$$

$$= \frac{\mathbb{E}[DY \mid X \approx c] - \mathbb{E}[(1-D)Y \mid X \approx c] - \tau_0 \mathbb{E}[Y \mid X \approx c, M = 1]}{\mathbb{E}[D \mid X \approx c]}$$
$$= \frac{\mathbb{E}[(2D-1)Y \mid X \approx c] - \tau_0 \mathbb{E}[Y \mid X \approx c, M = 1]}{\tau_0 + (1-\tau_0)/2}$$

The CLATE is immediate. □

Proof of Lemma 2. We begin with $Y(1)$. Since $\mathbb{Q}$ produces the observable distribution of $Y(1), D = 1 \mid X \approx c$, we have:

$$\begin{aligned}
\mathbb{Q}(X > c \mid X \approx c) d\mathbb{Q}(Y(1) \mid X = c^+) &= d\mathbb{Q}(Y(1) \mid X \approx c) \mathbb{Q}(X > c \mid X \approx c, Y(1)) \\
&= d\mathbb{Q}(Y(1) \mid X \approx c) \left(q_1(Y(1)) + (1 - q_1(Y(1)))/2\right) \\
d\mathbb{Q}(Y(1) \mid X \approx c) &= \frac{2\mathbb{P}(X > c \mid X \approx c)}{1 + q_1(Y(1))} d\mathbb{Q}(Y(1) \mid X \approx c)
\end{aligned}$$

We can also derive the probability of a treated observation being manipulated under $\mathbb{Q}$ through Bayes' Rule:

$$\begin{aligned}
\mathbb{Q}(M = 0 \mid Y(1), X = c^+) &= \frac{d\mathbb{Q}(Y(1) \mid X \approx c) \mathbb{Q}(M = 0 \mid X \approx c, Y(1)) \mathbb{Q}(X > c \mid X \approx c, Y(1), M = 0)}{\mathbb{Q}(X > c \mid X \approx c) d\mathbb{Q}(Y(1) \mid X = c^+)} \\
&= \frac{d\mathbb{Q}(Y(1) \mid X \approx c) * (1 - q_1(Y(1)))/2}{d\mathbb{Q}(Y(1) \mid X \approx c)(q_1(Y(1)) + (1 - q_1(Y(1)))/2)} \\
&= \frac{1 - q_1(Y(1))}{1 + q_1(Y(1))}
\end{aligned}$$

As a result:

$$\begin{aligned}
d\mathbb{Q}(Y(1) \mid X \approx c, M = 0) &= d\mathbb{Q}(Y(1) \mid X = c^+, M = 0) \\
&= \frac{d\mathbb{Q}(Y(1) \mid X = c^+) \mathbb{Q}(M = 0 \mid X = c^+, Y(1))}{\mathbb{Q}(M = 0 \mid X = c^+)} \\
&= \frac{\frac{1 - q_1(Y(1))}{1 + q_1(Y(1))}}{1 - \tau} d\mathbb{P}(Y(1) \mid X = c^+)
\end{aligned}$$

This is our first equality.

We now turn our attention to $Y(0)$. By a similar observed-untreated-outcome argument, we have:

$$\begin{aligned}
&\mathbb{Q}(X < c \mid X \approx c) d\mathbb{Q}(Y(0) \mid X = c^-) \\
=& d\mathbb{Q}(Y(0) \mid X \approx c) \mathbb{Q}(X < c \mid X \approx c, Y(0)) \\
=& d\mathbb{Q}(Y(0) \mid X \approx c) \mathbb{Q}(M = 0 \mid X \approx c, Y(0)) \mathbb{Q}(X < c \mid X \approx c, Y(0), M = 0) \\
=& d\mathbb{Q}(Y(0) \mid X \approx c)(1 - q_0(Y(0)))/2
\end{aligned}$$

Since $\mathbb{Q}(X < c \mid X \approx c) = \mathbb{P}(X < c \mid X \approx c) = \frac{1-\tau_0}{2}$, we can then obtain:

$$d\mathbb{Q}(Y(0) \mid X \approx c) = \frac{1-\tau_0}{1-q_0(Y(0))} d\mathbb{Q}(Y(0) \mid X = c^-)$$

We can also split up $d\mathbb{Q}(Y(0) \mid X \approx c)$ as:

$$\begin{aligned} d\mathbb{Q}(Y(0) \mid X \approx c) &= \tau_0 d\mathbb{Q}(Y(0) \mid X \approx c, M = 1) + (1-\tau_0) d\mathbb{Q}(Y(0) \mid X \approx c, M = 0) \\ &= \tau_0 d\mathbb{Q}(Y(0) \mid X \approx c, M = 1) + (1-\tau_0) d\mathbb{Q}(Y(0) \mid X = c^-) \end{aligned}$$

So that we can combine terms to obtain:

$$\begin{aligned} d\mathbb{Q}(Y(0) \mid X \approx c, M = 1) &= \frac{1-\tau_0}{\tau_0} \frac{q_0(Y(0))}{1-q_0(Y(0))} d\mathbb{Q}(Y(0) \mid X = c^-) \\ \frac{1-\tau_0}{\tau_0} &= \frac{2(1-\tau)}{\tau} \\ d\mathbb{Q}(Y(0) \mid X \approx c, M = 1) &= \frac{2(1-\tau)}{\tau} \frac{q_0(Y(0))}{1-q_0(Y(0))} d\mathbb{Q}(Y(0) \mid X = c^-) \end{aligned}$$

Which is the final equality after substituting in $d\mathbb{Q}(Y(0) \mid X = c^-) = d\mathbb{P}(Y(0) \mid X = c^-)$. □

Proof of Proposition 1. With (9), the expressions for $\lambda(R), \lambda_{True}(R)$ and W are straightforward. $E[W] = 1$ is also trivial. Consider general $[l, u]$ bounds:

$$\frac{q_1/\left(1-q_1\right)}{\tau/\left(2\left(1-\tau\right)\right)} \in [l, u]$$

We want to put bounds on W, and hence obtain $\underline{w}$ and $\bar{w}$. Note that there are no covariates here, so R is inconsequential.

$$l \leq \frac{q_1 2\left(1-\tau\right)}{\tau\left(1-q_1\right)} \leq u$$

By manipulating the W expression,

$$\begin{aligned} \frac{1}{W} &= \left(1-\tau\right) \frac{1+q_1}{1-q_1} = \left(1-\tau\right)\left(1 + \frac{2q_1}{1-q_1}\right) \\ \frac{2q_1}{1-q_1} &= \frac{1}{W\left(1-\tau\right)} - 1 \end{aligned}$$

Then, the sensitivity assumption becomes:

$$\begin{aligned} l \leq \left(\frac{1}{W\left(1-\tau\right)} - 1\right) \frac{1-\tau}{\tau} \leq u \\ \frac{\tau l}{1-\tau} \leq \frac{1}{W\left(1-\tau\right)} - 1 \leq \frac{\tau u}{1-\tau} \end{aligned}$$

Taking the lower bound first,

$$\frac{1}{W\left(1-\tau\right)} \geq 1 + \frac{\tau l}{1-\tau}$$

$$
\begin{aligned}
&= \frac{1-\tau+\tau l}{1-\tau} \\
\frac{1}{W} &\geq 1-\tau+\tau l \\
W &\leq \frac{1}{1-\tau+\tau l}
\end{aligned}
$$

Using an analogous argument on the other side, we have:

$$
\frac{1}{1-\tau+\tau u} \leq W \leq \frac{1}{1-\tau+\tau l}
$$

To complete the argument, substitute $u = \Lambda_1^+$ and $l = \Lambda_1^-$. ☐

Proof of Proposition 2. The expressions are immediate. ☐

Proof of Proposition 3.

$$
\begin{aligned}
\psi &= \frac{1-\tau}{1-2\tau} E\left[Y(1)|M=0, X \approx c\right] - \frac{\tau}{2-\tau} E\left[Y(0)|M=1, X \approx c\right] \\
&= \frac{1}{1-2\tau} E\left[Y \frac{1-q_1}{1+q_1} \frac{1\left\{X=c^+\right\}}{P\left(X=c^+\right)}\right] - \frac{2\left(1-\tau\right)}{2-\tau} E\left[Y \frac{q_0}{1-q_0} \frac{1\left\{X=c^-\right\}}{P\left(X=c^-\right)}\right] \\
&= E\left[\left(\frac{1}{1-2\tau} \frac{1-q_1}{1+q_1} \frac{1\left\{X=c^+\right\}}{P\left(X=c^+\right)} - \frac{2\left(1-\tau\right)}{2-\tau} \frac{q_0}{1-q_0} \frac{1\left\{X=c^-\right\}}{P\left(X=c^-\right)}\right) Y\right]
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\lambda_{True}(R) &= \frac{1}{1-2\tau} \frac{1-q_1}{1+q_1} \frac{1\left\{X=c^+\right\}}{P\left(X=c^+\right)} - \frac{2\left(1-\tau\right)}{2-\tau} \frac{q_0}{1-q_0} \frac{1\left\{X=c^-\right\}}{P\left(X=c^-\right)} \\
\lambda(R) &= \frac{1-\tau}{1-2\tau} \frac{1\left\{X=c^+\right\}}{P\left(X=c^+\right)} - \frac{\tau}{2-\tau} \frac{1\left\{X=c^-\right\}}{P\left(X=c^-\right)} \\
&= \frac{1\left\{X=c^+\right\}\left(2-\tau\right)\left(1-\tau\right)P\left(X=c^-\right) - \tau 1\left\{X=c^-\right\}\left(1-2\tau\right)P\left(X=c^+\right)}{\left(1-2\tau\right)\left(2-\tau\right)P\left(X=c^+\right)P\left(X=c^-\right)}
\end{aligned}
$$

Since $W = \frac{\lambda_{True}(R)}{\lambda(R)}$,

$$
\begin{aligned}
W = &\frac{1}{1-2\tau} \frac{1-q_1}{1+q_1} \frac{1\left\{X=c^+\right\}}{P\left(X=c^+\right)} \frac{\left(1-2\tau\right)\left(2-\tau\right)P\left(X=c^+\right)P\left(X=c^-\right)}{1\left\{X=c^+\right\}\left(2-\tau\right)\left(1-\tau\right)P\left(X=c^-\right) - \tau 1\left\{X=c^-\right\}\left(1-2\tau\right)P\left(X=c^+\right)} \\
&- \frac{2\left(1-\tau\right)}{2-\tau} \frac{q_0}{1-q_0} \frac{1\left\{X=c^-\right\}}{P\left(X=c^-\right)} \frac{\left(1-2\tau\right)\left(2-\tau\right)P\left(X=c^+\right)P\left(X=c^-\right)}{1\left\{X=c^+\right\}\left(2-\tau\right)\left(1-\tau\right)P\left(X=c^-\right) - \tau 1\left\{X=c^-\right\}\left(1-2\tau\right)P\left(X=c^+\right)} \\
= &\frac{1-q_1}{1+q_1} 1\left\{X=c^+\right\} \frac{\left(2-\tau\right)P\left(X=c^-\right)}{1\left\{X=c^+\right\}\left(2-\tau\right)\left(1-\tau\right)P\left(X=c^-\right) - \tau 1\left\{X=c^-\right\}\left(1-2\tau\right)P\left(X=c^+\right)} \\
&- 2\left(1-\tau\right) \frac{q_0}{1-q_0} 1\left\{X=c^-\right\} \frac{\left(1-2\tau\right)P\left(X=c^+\right)}{1\left\{X=c^+\right\}\left(2-\tau\right)\left(1-\tau\right)P\left(X=c^-\right) - \tau 1\left\{X=c^-\right\}\left(1-2\tau\right)P\left(X=c^+\right)} \\
= &\frac{1-q_1}{1+q_1} 1\left\{X=c^+\right\} \frac{\left(2-\tau\right)P\left(X=c^-\right)}{\left(2-\tau\right)\left(1-\tau\right)P\left(X=c^-\right)} - 2\left(1-\tau\right) \frac{q_0}{1-q_0} 1\left\{X=c^-\right\} \frac{\left(1-2\tau\right)P\left(X=c^+\right)}{-\tau\left(1-2\tau\right)P\left(X=c^+\right)} \\
= &\frac{1-q_1}{1+q_1} 1\left\{X=c^+\right\} \frac{1}{\left(1-\tau\right)} + 2\left(1-\tau\right) \frac{q_0}{1-q_0} 1\left\{X=c^-\right\} \frac{1}{\tau}
\end{aligned}
$$

Hence,

$$W = \frac{1-q_1}{1+q_1} 1\left\{X = c^+\right\} \frac{1}{(1-\tau)} + \frac{2(1-\tau)}{\tau} \frac{q_0}{1-q_0} 1\left\{X = c^-\right\}$$

Then, it can be verified that $E[W] = 1$:

$$\begin{aligned}
E[W] &= E\left[\frac{1-q_1}{1+q_1} 1\left\{X = c^+\right\} \frac{1}{(1-\tau)} + 2(1-\tau) \frac{q_0}{1-q_0} 1\left\{X = c^-\right\} \frac{1}{\tau}\right] \\
&= (1-\tau) \frac{1}{(1-\tau)} P\left(X = c^+\right) + \frac{2(1-\tau)}{\tau} P\left(X = c^-\right) \frac{\tau}{2(1-\tau)} \\
&= P\left(X = c^+\right) + P\left(X = c^-\right) = 1
\end{aligned}$$

For CATE, we don't need the same q to work on both bounds.

$$W = \frac{1-q_1}{1+q_1} 1\left\{X = c^+\right\} \frac{1}{(1-\tau)} + \frac{2(1-\tau)}{\tau} \frac{q_0}{1-q_0} 1\left\{X = c^-\right\}$$

Sensitivity assumption is:

$$l \leq \frac{2q(1-\tau)}{\tau(1-q)} \leq u$$

Then, combining the two optimization problems from Proposition 1 and Proposition 2,

$$1\left\{X = c^+\right\} \frac{1}{1-\tau+\tau u} + 1\left\{X = c^-\right\} l \leq W \leq 1\left\{X = c^+\right\} \frac{1}{1-\tau+\tau l} + 1\left\{X = c^-\right\} u$$

□

Proof of Proposition 4. Due to (12),

$$l_1(X) \leq \frac{1-e(X)}{\frac{e(X)}{e_1(Z,Y(1))}(1-e_1(Z,Y(1)))} \leq u_1(X)$$

$$l_1(X) \leq \frac{1-e(X)}{\frac{e(X)}{e_1(Z,Y(1))} - e(X)} \leq u_1(X)$$

$$l_1(X) \leq \frac{1-e(X)}{W-e(X)} \leq u_1(X)$$

Observe that $W > e(X)$. If $W < e(X)$, then $(W-e(X))\, l_1(X) \geq 1-e(X) \geq u_1(X)\,(W-e(X))$. Since $e(X) \in (0,1)$ and $l_1(X) > 0$, $1-e(X) \leq (W-e(X))\, l_1(X) < 0$ is a contradiction. Hence

$$(W-e(X))\, l_1(X) \leq 1-e(X)$$

$$W \leq \frac{1}{l_1(X)}(1-e(X)) + e(X)$$

□

Proof of Proposition 5. The form of λ_{True}, $\lambda(R)$, and W follow by a simple adaptation of the proof of Proposition 4. It only remains to verify the form of $\underline{w}$ and $\bar{w}$. The likelihood ratio bounds for treated observations $Z\underline{w}(R)$ and $Z\underline{w}(R)$ follow by Proposition 4, so it only remains to verify the form of $(1-Z)\underline{w}(R)$

and $(1-Z)\underline{w}(R)$ where $(1-Z)\underline{w}(R) \le \frac{(1-Z)(1-e(X))}{1-e_0(X,Y(0))} \le (1-Z)\bar{w}(R)$ under the assumption $l_0(X) \le \frac{e_0(X,Y(0))/(1-e_0(X,Y(0)))}{e(X)/(1-e(X))} \le u_0(X)$.

We rewrite the $Y(0)$ sensitivity assumption as $u_0^{-1} \le \frac{(1-e_0(X,Y(0)))/e_0(X,Y(0))}{(1-e(X))/e(X)} \le l_0^{-1}$. Then by an analogous version of the proof of Proposition 4, we have:

$$(1-Z)\underline{w}(R) = (1-Z)\left(1-e(X)+e(X)l_0\right)$$
$$(1-Z)\bar{w}(R) = (1-Z)\left(1-e(X)+e(X)u_0\right),$$

completing the proof. □

A.4 Proofs for Additional Claims

Proof of Proposition 6. We show that the upper bound is at most $E[\lambda(R)\mu(R)]$ plus one-half the proposed width; the lower bound follows symmetrically.

Note from Theorem 2 that the upper bound for the identified set is generally:

$$\psi^+ = E\left[\lambda(R)Y + (1-\underline{w}(R))\left(\rho(R)-\lambda(R)\mu(R)\right)\right],$$

where

$$\rho(R) - \lambda(R)\mu(R) = E[\lambda(R)Y \mid R, \lambda(R)Y > Q_{\tau(R)}(\lambda(R)Y \mid R)] - E[\lambda(R)Y \mid R]$$

is the difference between the conditional value at risk of $\lambda(R)Y$ and the conditional mean of $\lambda(R)Y$. (If $\tau(R)=1$, then $\rho(R) = Q_1(\lambda(R)Y \mid R)$.)

In this example with conditionally normal outcomes, the result can be stated as:

$$\begin{aligned}\psi^+ &= \mathbb{E}\left[\lambda(R)Y + (1-\underline{w}(R))\sigma(R)\lambda(R)\frac{\phi(q_{\tau(R)})}{1-\tau(R)}\right] \\ &= \mathbb{E}\left[\lambda(R)\mu(R) + (1-\underline{w}(R))\sigma(R)\lambda(R)\frac{\phi(q_{\tau(R)})}{1-\tau(R)}\right],\end{aligned}$$

where q_τ is the τ^{th} quantile of a standard normal distribution. By existing arguments (e.g. Pinelis (2019)), the inverse mills ratio $\phi(q)/(1-\Phi(q))$ has the upper bound $\sqrt{2/\pi}+q$. Therefore the APO upper bound can be further bounded as:

$$\psi^+ \le E[\lambda(R)\mu(R)] + E[(1-\underline{w}(R))\sigma(R)\lambda(R)\left(\sqrt{2/\pi}+q_{\tau(R)}\right)].$$

It remains to bound $q_{\tau(R)}$. By standard arguments, if $S \sim N(0,1)$, then $P(S > s) \le exp(-s^2/2)$. We substitute $s = q_{\tau(R)}$ to obtain:

$$\begin{aligned}1-\tau(R) = P(S > q_{\tau(R)}) &\le exp(-q_{\tau(R)}^2/2) \\ \log(1-\tau(R)) &\le -q_{\tau(R)}^2/2 \\ \sqrt{\log\left(\frac{1}{1-\tau(R))^2}\right)} &\ge q_{\tau(R)}.\end{aligned}$$

Therefore we have bounded the identified set as:

$$\psi^+ \leq E\left[\lambda(R)\mu(R) + (1-\underline{w}(R))\sigma(R)\lambda(R)\left(\sqrt{2/\pi} + \sqrt{2(1-\underline{w}(R))^2\log\left(\frac{1}{1-\tau(R))}\right)}\right)\right].$$

Now we bound the second square root, using the identity:

$$\frac{1}{1-\tau(R)} = \frac{\bar{w}(R)-\underline{w}(R)}{1-\underline{w}(R)} = 1 + \frac{\bar{w}(R)-1}{1-\underline{w}(R)}.$$

Therefore:

$$\begin{aligned}2(1-\underline{w}(R))^2\log\left(\frac{1}{1-\tau(R))}\right) &= 2(1-\underline{w}(R))^2\log(\bar{w}(R)-\underline{w}(R)) - 2(1-\underline{w}(R))^{2-2\epsilon}(1-\underline{w}(R))^{2\epsilon}\log(1-\underline{w}(R)) \\ &\leq 2(1-\underline{w}(R))^2\log(\bar{w}(R)) + \frac{(1-\underline{w}(R))^{2-2\epsilon}}{e*\epsilon}.\end{aligned}$$

So that we now have the bound:

$$\begin{aligned}\psi^+ &\leq E\left[\lambda(R)\mu(R)\right] + E\left[\sigma(R)\lambda(R)\left((1-\underline{w}(R))\sqrt{2/\pi} + \sqrt{2(1-\underline{w}(R))^2\log(\bar{w}(R)) + \frac{(1-\underline{w}(R))^{2-2\epsilon}}{e*\epsilon}}\right)\right] \\ &\leq E\left[\lambda(R)\mu(R)\right] + E\left[\sigma(R)\lambda(R)(1-\underline{w}(R))\left(\sqrt{2\log(\bar{w}(R))} + \sqrt{2/\pi} + (1-\underline{w}(R))^{\epsilon}\sqrt{1/(e*\epsilon)}\right)\right].\end{aligned}$$

Applying the same argument to the symmetric lower bound completes the proof. □

Proof of Corollary 1. We first show the identified set of $E[Y(1)]$ is unbounded if $c > \eta_1$; the argument is symmetric for the lower bound of $E[Y(0)]$ if $c > \eta_2$.

By the decomposition in the proof of Proposition 6, the upper bound of the identified set of $E[Y(1)]$ is:

$$\psi^+ = E\left[\lambda(R)\mu(R) + (1-\underline{w}(R))\left(\rho(R) - \lambda(R)\mu(R)\right)\right],$$

where $\lambda(R) = Z/e(X)$, $\underline{w}(R) = Z\frac{e(X)}{e(X)+c}$, and $\mu(R) = E[Y \mid R]$. We can lower bound the upper bound as:

$$\begin{aligned}\psi^+ &\geq E\left[\lambda(R)\mu(R) + I(e(X) < [\eta_1, c])(1-\underline{w}(R))\left(\rho(R) - \lambda(R)\mu(R)\right)\right] \\ &\geq E\left[\lambda(R)\mu(R) + I(e(X) \in [\eta_1, c])\frac{1}{2}\frac{Z}{e(X)}E[Y - \mu(R) \mid Y, R \geq Q_1(Y \mid R)]\right] \\ &\geq E\left[\lambda(R)\mu(R) + I(e(X) \in [\eta_1, c])\frac{1}{2}\frac{\eta_1}{c} * \infty\right],\end{aligned}$$

where the infinite conditional value at risk happens for all X with $\sigma(X,1) > 0$, which happens almost surely by assumption. Since $\eta_1 = \inf p \mid P(e(X) > p) > 0$ by definition and $\eta_1 > 0$ by assumption, $E[I(e(X) \in [\eta_1, c])\frac{1}{2}\frac{\eta_1}{c} \mid R] > 0$ so that the identified set would be unbounded.

Now suppose $c < \eta_1$ and we wish to show that the identified set is uniformly bounded. Write D as the upper bound of the support of $\frac{\sigma(X,Z)}{e(X)} + \frac{\sigma(X,Z)}{1-e(X)}$, which by assumption is finite. Recall by Proposition 6 that the identified set can be upper bounded as follows:

$$\begin{aligned}\psi^+ &\leq E[\lambda(R)Y] + E\left[\sigma(R)\lambda(R)(1-\underline{w}(R))\left(\sqrt{2\log(\bar{w}(R))} + \sqrt{2/\pi} + (1-\underline{w}(R))^{\epsilon}\sqrt{1/(e*\epsilon)}\right)\right] \\ &\leq E[\lambda(R)Y] + DE\left[Z\sqrt{2\log(\bar{w}(R))} + \sqrt{2/\pi} + 1/e\right].\end{aligned}$$

It is clear it only remains to bound $\sqrt{2}E[Z\log(\bar{w}(R))]$, where $\bar{w}(R) = Ze(X)/(e(X)-c) + (1-Z)(1-e(X))/(1-e(X)-c)$. Suppose the propensity density is $f_{e(X)}(p)$ and is upper bounded by $\bar{f}_{e(X)}$:

$$\begin{aligned}
E[Z\log(\bar{w}(R))] &= \int_{\eta_1}^{1-\eta_2} p\log(p/(p-c))f_{e(X)}(p) \\
&\leq \bar{f}_{e(X)} \int_{\eta_1}^{1-\eta_2} log(p/(p-\eta_1))dp \leq -\bar{f}_{e(X)} \int_{\eta_1}^{1-\eta_2} \log(p-\eta_1)dp \\
&\leq -\bar{f}_{e(X)} \int_0^1 log(t)dt = \bar{f}_{e(X)}.
\end{aligned}$$

Therefore an APO upper bound is:

$$\begin{aligned}
\psi^+ &\leq E[\lambda(R)Y] + E\left[\sigma(R)\lambda(R)(1-\underline{w}(R))\left(\sqrt{2\log(\overline{w}(R))} + \sqrt{2/\pi} + (1-\underline{w}(R))^\epsilon\sqrt{1/(e*\epsilon)}\right)\right] \\
&\leq E[\lambda(R)Y] + DE\left[\sqrt{2}\bar{f}_{e(X)} + \sqrt{2/\pi} + 1/e\right].
\end{aligned}$$

Since this bound holds symmetrically for the lower bound of $E[Y(0)]$, writing $B = 2DE\left[\sqrt{2}\bar{f}_{e(X)} + \sqrt{2/\pi} + 1/e\right]$ completes the proof. □